

LOCAL SUPREMA OF DIRICHLET POLYNOMIALS AND ZEROFREE REGIONS OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. A new zerofree region of the Riemann Zeta-function ζ is identified by using Turán's localization criterion linking zeros of ζ with uniform local suprema of sets of Dirichlet polynomials expanded over the primes. The proof is based on a randomization argument. An estimate for local extrema for some finite families of shifted Dirichlet polynomials, is established by preliminary considering their local increment properties, by means of Montgomery-Vaughan's variant of Hilbert's inequality. A covering argument combined with Turán's localization criterion allows to conclude.

2010 AMS Mathematical Subject Classification: Primary: 11M26 ; Secondary: 26D05, 60G17.

Keywords and phrases: Riemann zeta function, zeros, Dirichlet polynomials, Turán's localization criterion, increments, local suprema.

1. Main Result

The question of the existence of an eventual explicit relation between the zeros of the Riemann Zeta function $\zeta(s)$, $s = \sigma + it$ and the prime numbers was raised already by Landau in [1]. Motivated by Landau's remark, Turán had much investigated the connection between zerofree regions of ζ and local bounds of Dirichlet polynomials expanded over the primes, see [5] and [6], Chapters 33-36. Among the several strong localization results stated in [5], the following semi-global criterion (Theorem 3') is of particular relevance in the present work.

Turán's Localization Criterion. *Let D be some positive real and $0 < E \leq 9/10$. Suppose there exist positive reals T, β , $0 < \beta < 1$ such that for $T - T^E \leq \tau \leq T + T^E$, the inequality*

$$(1.1) \quad \left| \sum_{N_1 \leq p \leq N_2} p^{-i\tau} \right| \leq c \frac{N \log^{10} N}{\tau^\beta},$$

holds for

$$T^{D(1-\beta^{1/6})} \leq N \leq N_1 < N_2 \leq 2N \leq T^{D(1+\beta^{1/6})}$$

where c stands for positive numerical, explicitly calculable constant.

Then $\zeta(s) \neq 0$ in the parallelogram $\sigma > 1 - \beta^2$, $T - T^E \leq t \leq T + T^E$.

In this article, we show by using a local randomization argument, that Turán's approach for localizing zeros of ζ is sufficiently powerful to permit to identify a completely new semi-global zerofree region.

Our main result states:

Theorem 1.1. *Let $0 < \alpha^* < 1$. There exist $1/2 < \sigma_0 < 1$, $B \geq 4$, $\nu_0 < \infty$, such that: For all $\nu \geq \nu_0$, there exists at least $\alpha^* 2^{B\nu+1}$ indices j for which*

$$\zeta(\sigma + it) \neq 0 \quad \forall \sigma \geq \sigma_0, \quad \forall t \in [2^{2B\nu} + (j-1)2^{B\nu-1}, 2^{2B\nu} + j2^{B\nu-1}].$$

It follows from the proof that any value $\sigma_0 > 1 - 1/(19)^{12}$ is for instance suitable. The same approach permits to get only slightly better thresholds.

In order to bound $|\sum_{N_1 \leq p \leq N_2} p^{-i\tau}|$, uniformly over a family of suitable segments $[N_1, N_2]$ of the real line, we use an approach which can be described as follows. Let $\varphi_1, \dots, \varphi_N$ be distinct reals. Consider a finite family of Dirichlet polynomials $P^s(t) = \sum_{n=1}^N c_n^s e^{it\varphi_n}$, $s \in S$, c_1^s, \dots, c_N^s being complex numbers. Instead of directly searching a bound of $\sup_S |P^s(t)|$, uniformly in t over some finite interval L , we operate with the *shifted* Dirichlet polynomials

$$(1.2) \quad P_\theta^s(t) = \sum_{n=1}^N c_n^s e^{i(\theta+t)\varphi_n},$$

where θ belongs to some fixed interval J , and θ will be treated as a random parameter. Given some interval L , $\{P_\theta^s(t), s \in S, t \in L, \theta \in J\}$ is considered at some intermediate stage of the proof, as a random process built on J , of which we estimate the increments by means of variant form of Hilbert's inequality due to Montgomery and Vaughan. A classical argument from random processes machinery, allows to efficiently control suprema, namely here $\sup_{t \in L} \sup_S |P_\theta^s(t)|$.

Another step is devoted to carefully adjusting some inherent family of parameters, in order to apply Turán's result. Once this is achieved, a family of intervals $(I_\theta)_\theta$ free of zeros is then exhibited. The family is indexed by a measurable set of θ 's of controllable positive measure. Finally, a covering argument allows to establish the existence of a semi-global region. This is the strategy we apply.

2. Local Mean Value Results

Let q be some positive integer and denote

$$E_q = \left\{ \underline{k} = (k_1, \dots, k_N); k_i \in \mathbb{N} : k_1 + \dots + k_N = q \right\}.$$

Let $\varphi_1, \dots, \varphi_N$ be linearly independent reals. Introduce a *coefficient of linear spacing* of order q by putting

$$\xi_\varphi(N, q) = \inf_{\substack{\underline{h}, \underline{k} \in E_q \\ \underline{h} \neq \underline{k}}} |(h_1 - k_1)\varphi_1 + \dots + (h_N - k_N)\varphi_N|.$$

By assumption $\xi_\varphi(N, q) > 0$ and $\xi_\varphi(N, 1) = \inf\{|\varphi_i - \varphi_j| : i \neq j\}$. In the case $\varphi_n = \log p_n$, p_n denoting the n -th consecutive prime, we have the classical estimate $\xi_\varphi(N, q) \geq p_N^{-q}$, see before (2.13) for a proof.

We estimate the local increments of P , defined in (1.2). Let J be a bounded interval and let $|J|$ denote its length. Let m_J denote the normalized Lebesgue measure on J . With the notation (1.2), if $J = [a, b]$ then $\|P(t) - P(s)\|_{m_J, 2q}$ and $\|P(t)\|_{m_J, 2q}$ respectively denote

$$\left(\frac{1}{b-a} \int_a^b |P(\theta+t) - P(\theta+s)|^{2q} d\theta \right)^{1/2q}, \quad \left(\frac{1}{b-a} \int_a^b |P(\theta+t)|^{2q} d\theta \right)^{1/2q}.$$

Introduce the stationary metric on the real line defined by

$$d(s, t) = d_N(s, t) := \left(2 \sum_{n=1}^N |c_n|^2 \left| \sin \frac{(t-s)\varphi_n}{2} \right|^2 \right)^{1/2}.$$

Proposition 2.1. *a) For any reals s and t ,*

$$\|P(t) - P(s)\|_{m_J, 2q} \leq \left(q! + \frac{2 \min(N^q, \pi q!)}{|J|\xi} \right)^{1/2q} d(s, t).$$

And

$$\|P(t)\|_{m_J, 2q} \leq \left(q! + \frac{2 \min(N^q, \pi q!)}{T\xi}\right)^{1/2q} \left(\sum_{n=1}^N |c_n|^2\right)^{1/2}.$$

By taking $J = [-T, T]$, $t = 0$ in the last estimate, we deduce

Corollary 2.2. *We have the following bound*

$$\frac{1}{2T} \int_{-T}^T \left| \sum_{n=1}^N c_n e^{i\theta \varphi_n} \right|^{2q} d\theta \leq q! \left(1 + \frac{2\pi}{T\xi_\varphi(N, q)}\right) \left(\sum_{n=1}^N |c_n|^2\right)^q.$$

In particular,

$$\frac{1}{2T} \int_{-T}^T \left| \sum_{p \leq N} \frac{c_p}{p^{i\theta}} \right|^{2q} d\theta \leq q! \left(1 + \frac{2\pi N^q}{T}\right) \left(\sum_{p \leq N} |c_p|^2\right)^q.$$

Now put

$$\mathcal{B} = \mathcal{B}_\varphi(J, N, q) = \left[q! \left(1 + \frac{2\pi}{|J|\xi_\varphi(N, q)}\right) \right]^{1/2q}.$$

Theorem 2.3. *Let $\tilde{\varphi}_N = \sup_{n \leq N} |\varphi_n|$. There exists a constant C_q depending on q only, such that for any interval \bar{L} ,*

$$\begin{aligned} \left\| \sup_{t \in \bar{L}} |P(t)| \right\|_{m_J, 2q} &\leq C_q \mathcal{B} \max \left\{ 1, |L| \tilde{\varphi}_N \right\}^{1/2q} \left\{ \left[\sum_{n=1}^N |c_n|^2 \right]^{1/2} + \right. \\ &\quad \left. \min \left(|L|, \frac{1}{\tilde{\varphi}_N} \right) \left[\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right]^{1/2} \right\}. \end{aligned}$$

Proof of Proposition 2.1. Let $J = [d, d + T]$. Write more shortly $\xi = \xi_\varphi(N, q)$. Plainly

$$\begin{aligned} (P(\theta + t) - P(\theta + s))^q &= \left(\sum_{n=1}^N c_n e^{i\theta \varphi_n} (e^{it\varphi_n} - e^{is\varphi_n}) \right)^q \\ &= \sum_{\underline{k} \in E_q} \frac{q!}{k_1! \dots k_N!} \prod_{n=1}^N c_n^{k_n} e^{i\theta k_n \varphi_n} (e^{it\varphi_n} - e^{is\varphi_n})^{k_n} \end{aligned}$$

Put $\gamma_n = e^{it\varphi_n} - e^{is\varphi_n}$. Thus

$$\begin{aligned} &|P(\theta + t) - P(\theta + s)|^{2q} \\ &= \sum_{\underline{k}, \underline{h} \in E_q} \frac{(q!)^2}{k_1! h_1! \dots k_N! h_N!} \prod_{n=1}^N c_n^{k_n} \overline{c_n}^{h_n} e^{i\theta(k_n - h_n)\varphi_n} \gamma_n^{k_n} \overline{\gamma_n}^{h_n} \\ (2.1) \quad &= \sum_{\underline{k} \in E_q} \left(\frac{q!}{k_1! \dots k_N!} \right)^2 \prod_{n=1}^N (|c_n| |\gamma_n|)^{2k_n} + R(\theta) \end{aligned}$$

where

$$(2.2) \quad R(\theta) = \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \left(\frac{(q!)^2}{k_1! h_1! \dots k_N! h_N!} \right) \prod_{n=1}^N (c_n \gamma_n)^{k_n} (\overline{c_n} \overline{\gamma_n})^{h_n} e^{i\theta(k_n - h_n)\varphi_n}.$$

Owing to linear independence $\sum_{n=1}^N (k_n - h_n)\varphi_n = 0$, iff $k_n = h_n$, $n = 1, \dots, N$. By integrating

$$\frac{1}{T} \int_J |P(\theta + t) - P(\theta + s)|^{2q} d\theta = \sum_{\underline{k} \in E_q} \left(\frac{q!}{k_1! \dots k_N!} \right)^2 \prod_{n=1}^N (|c_n| |\gamma_n|)^{2k_n}$$

$$\begin{aligned}
& + \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{(q!)^2}{k_1! h_1! \dots k_N! h_N!} \prod_{n=1}^N (c_n \gamma_n)^{k_n} (\overline{c_n \gamma_n})^{h_n} \\
(2.3) \quad & \times \left[\frac{e^{i(d+T) \sum_{n=1}^N (k_n - h_n) \varphi_n} - e^{id \sum_{n=1}^N (k_n - h_n) \varphi_n}}{iT (\sum_{n=1}^N (k_n - h_n) \varphi_n)} \right].
\end{aligned}$$

Put

$$\mathbf{c}_{\underline{k}} = \prod_{n=1}^N \frac{(c_n \gamma_n e^{i(d+T) \varphi_n})^{k_n}}{k_n!}, \quad \mathbf{d}_{\underline{k}} = \prod_{n=1}^N \frac{(c_n \gamma_n e^{id \varphi_n})^{k_n}}{k_n!}, \quad \mathbf{l}_{\underline{k}} = \sum_{n=1}^N k_n \varphi_n.$$

Then

$$\begin{aligned}
& \frac{1}{T} \int_J |P(\theta + t) - P(\theta + s)|^{2q} d\theta \\
(2.4) \quad & = q!^2 \sum_{\underline{k} \in E_q} |\mathbf{d}_{\underline{k}}|^2 + \frac{(q!)^2}{iT} \left\{ \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{\mathbf{c}_{\underline{k}} \overline{\mathbf{c}_{\underline{h}}}}{\mathbf{l}_{\underline{k}} - \mathbf{l}_{\underline{h}}} - \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{\mathbf{d}_{\underline{k}} \overline{\mathbf{d}_{\underline{h}}}}{\mathbf{l}_{\underline{k}} - \mathbf{l}_{\underline{h}}} \right\}.
\end{aligned}$$

Each of the two claimed bounds will now be deduced from either Hilbert's inequality or Cauchy-Schwarz inequality. Recall Hilbert's inequality ([2], p.138): *Let $\lambda_1, \dots, \lambda_N$ be distinct real numbers, and suppose that $\delta > 0$ is chosen so that $|\lambda_m - \lambda_n| \geq \delta$ whenever $n \neq m$. Then*

$$(2.5) \quad \left| \sum_{\substack{1 \leq m, n \leq N \\ n \neq m}} \frac{x_m y_n}{\lambda_m - \lambda_n} \right| \leq \frac{\pi}{\delta} \left(\sum_{m=1}^N |x_m|^2 \right)^{1/2} \left(\sum_{n=1}^N |y_n|^2 \right)^{1/2}.$$

We shall apply it under the following form: let $\{x_{\underline{k}}, y_{\underline{k}}, \underline{k} \in E_q\}$. Let also $\{\lambda_{\underline{k}}, \underline{k} \in E_q\}$ be distinct real numbers such that $\min\{|\lambda_{\underline{k}} - \lambda_{\underline{h}}|, \underline{k} \neq \underline{h}\} \geq \delta$. Let $\nu = \# \{E_q\}$ and consider a bijection $i : \{1, \dots, \nu\} \rightarrow E_q$. By using (2.5)

$$\begin{aligned}
& \left| \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{x_{\underline{k}} y_{\underline{h}}}{\lambda_{\underline{k}} - \lambda_{\underline{h}}} \right| = \left| \sum_{\substack{1 \leq u, v \leq \nu \\ u \neq v}} \frac{x_{i(u)} y_{i(v)}}{\lambda_{i(u)} - \lambda_{i(v)}} \right| \\
& \leq \frac{\pi}{\delta} \left(\sum_{1 \leq u \leq \nu} |x_{i(u)}|^2 \right)^{1/2} \left(\sum_{1 \leq v \leq \nu} |y_{i(v)}|^2 \right)^{1/2} \\
(2.6) \quad & = \frac{\pi}{\delta} \left(\sum_{\underline{k} \in E_q} |x_{\underline{k}}|^2 \right)^{1/2} \left(\sum_{\underline{h} \in E_q} |y_{\underline{h}}|^2 \right)^{1/2}.
\end{aligned}$$

By applying Hilbert's inequality to each of the two sums in parenthesis of the right-term in (2.4), we obtain

$$(2.7) \quad \frac{(q!)^2}{T} \left| \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{\mathbf{c}_{\underline{k}} \overline{\mathbf{c}_{\underline{h}}}}{\mathbf{l}_{\underline{k}} - \mathbf{l}_{\underline{h}}} - \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{\mathbf{d}_{\underline{k}} \overline{\mathbf{d}_{\underline{h}}}}{\mathbf{l}_{\underline{k}} - \mathbf{l}_{\underline{h}}} \right| \leq \frac{2\pi(q!)^2}{T\xi} \sum_{\underline{k} \in E_q} |\mathbf{d}_{\underline{k}}|^2 \leq \frac{2\pi q!}{T\xi} d(s, t)^{2q},$$

since

$$\begin{aligned}
(q!)^2 \sum_{\underline{k} \in E_q} |\mathbf{d}_{\underline{k}}|^2 & = \sum_{k_1 + \dots + k_N = q} \left[\frac{q!}{k_1! \dots k_N!} \right]^2 \prod_{n=1}^N |c_n \gamma_n|^{2k_n} \\
& \leq q! \sum_{k_1 + \dots + k_N = q} \frac{q!}{k_1! \dots k_N!} \prod_{n=1}^N |c_n \gamma_n|^{2k_n} = q! \left[\sum_{n=1}^N |c_n \gamma_n|^2 \right]^q \\
& = q! \left[\sum_{n=1}^N |c_n|^2 |e^{it \varphi_n} - e^{is \varphi_n}|^2 \right]^q \\
(2.8) \quad & = q! \left[4 \sum_{n=1}^N |c_n|^2 \left| \sin \frac{(t-s) \varphi_n}{2} \right|^2 \right]^q = q! d(s, t)^{2q}.
\end{aligned}$$

Similarly as before

$$(2.9) \quad \sum_{\underline{k} \in E_q} \left(\frac{q!}{k_1! \dots k_N!} \right)^2 \prod_{n=1}^N |c_n|^{2k_n} |e^{it\varphi_n} - e^{is\varphi_n}|^{2k_n} \leq q! \left[\sum_{n=1}^N |c_n \gamma_n|^2 \right]^q = q! d(s, t)^{2q}.$$

By substituting in (2.4), we therefore get

$$(2.10) \quad \frac{1}{T} \int_J |P(\theta + t) - P(\theta + s)|^{2q} d\theta \leq q! \left(1 + \frac{2\pi}{T\xi} \right) d(s, t)^{2q}.$$

Without Hilbert's inequality, it is possible to arrive to a similar result. We have with (2.2), (2.8)

$$\begin{aligned} & \frac{1}{T} \int_J |P(\theta + t) - P(\theta + s)|^{2q} d\theta \leq q! d(s, t)^{2q} \\ & + \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{(q!)^2}{k_1! h_1! \dots k_N! h_N!} \prod_{n=1}^N (c_n \gamma_n)^{k_n} (\overline{c_n \gamma_n})^{h_n} \cdot \left| \frac{e^{iT \sum_{n=1}^N (k_n - h_n) \varphi_n} - 1}{iT (\sum_{n=1}^N (k_n - h_n) \varphi_n)} \right| \\ & \leq q! d(s, t)^{2q} + \frac{2}{T\xi} \left(2 \sum_{n=1}^N |c_n \sin \frac{(t-s)\varphi_n}{2}| \right)^q \left(2 \sum_{n=1}^N |c_n \sin \frac{(t-s)\varphi_n}{2}| \right)^q \\ & = q! d(s, t)^{2q} + \frac{2}{T\xi} \left(2 \sum_{n=1}^N |c_n \sin \frac{(t-s)\varphi_n}{2}| \right)^{2q} \\ & \leq \left(q! + \frac{2N^q}{T\xi} \right) d(s, t)^{2q}, \end{aligned}$$

where we used Cauchy-Schwarz inequality for getting the last estimate. Combining the two last estimates gives

$$(2.11) \quad \frac{1}{T} \int_J |P(\theta + t) - P(\theta + s)|^{2q} d\theta \leq \left(q! + \frac{2 \min(N^q, \pi q!)}{T\xi} \right) d(s, t)^{2q}.$$

Hence the first in assertion a). The same proof also yields, mutatis mutandis

$$(2.12) \quad \frac{1}{T} \int_J |P(\theta + s)|^{2q} d\theta \leq \left(\sum_{n=1}^N |c_n|^2 \right)^q \left(q! + \frac{2 \min(N^q, \pi q!)}{T\xi} \right).$$

We start with

$$P(\theta + t)^q = \left(\sum_{n=1}^N c_n e^{i\theta \varphi_n} e^{it\varphi_n} \right)^q = \sum_{\underline{k} \in E_q} \frac{q!}{k_1! \dots k_N!} \prod_{n=1}^N c_n^{k_n} e^{i\theta k_n \varphi_n} e^{it\varphi_n k_n}$$

and put this time $\gamma_n = e^{it\varphi_n}$. Then all calculations made after (2.1) remain valid. \square

Proof of Corollary 2.2. The first assertion is immediate. As for the second, we have to estimate

$$\xi_\varphi(N, q) = \inf_{\substack{\underline{h}, \underline{k} \in E_q \\ \underline{h} \neq \underline{k}}} |(h_1 - k_1)\varphi_1 + \dots + (h_N - k_N)\varphi_N|.$$

when $\varphi_n = \log p_n$. Let $\underline{\ell} = \underline{h} - \underline{k}$ and put

$$P^+ = \prod_{\ell_n > 0} p_n^{\ell_n}, \quad P^- = \prod_{\ell_n < 0} p_n^{-\ell_n}$$

Notice that $P^+ \neq P^-$ by assumption, and $\max(P^+, P^-) \leq p_N^q$. Suppose $P^+ > P^-$. Then

$$|\ell_1 \varphi_1 + \dots + \ell_N \varphi_N| = \left| \log \prod_{n=1}^N p_n^{\ell_n} \right| = \log \frac{P^+}{P^-} \geq \log 1 + \frac{1}{P^-} \geq \log 1 + \frac{1}{p_N^q} \geq \frac{1}{p_N^q}.$$

The case $P^+ < P^-$ is treated identically. Therefore

$$(2.13) \quad \xi_\varphi(N, q) \geq p_N^{-q}.$$

And so, it suffices to apply the first estimate to this case. \square

Proof of Theorem 2.3. We need some elements from the theory of stochastic processes. See [7], also [8] and references therein for a similar treatment. Let (T, δ) be a compact metric space and denote by D the diameter of T . For any $x \in T$ and $\varepsilon > 0$, let $B(x, \varepsilon)$ denote the open δ -ball of T with center x and radius ε . A stochastic process $X = \{X(t), t \in T\}$ is simply a collection of random variables indexed by T , and defined on some common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $1 < p < \infty$. Consider the increment condition

$$(2.14) \quad \|X(s) - X(t)\|_p \leq d(s, t) \quad (s, t \in T)$$

Assume that there exists a probability measure μ on T such that

$$(2.15) \quad \sup_{x \in T} \int_0^D \frac{d\varepsilon}{\mu(B(x, \varepsilon))^{1/p}} = M < \infty.$$

By Theorem 4.6 in [4], each separable process that satisfies the increment condition (2.14), is sample continuous. Moreover

$$(2.16) \quad \left\| \sup_{s, t \in T} |X(s) - X(t)| \right\|_p \leq K_p M,$$

where K_p depends on p only. The above inequality follows from the majorizing measure condition (2.15) and Proposition 2.7 in [4]. The sample continuity property is in turn obtained by combining Theorem 4.6 with Theorem 2.9 in [4]. A stochastic process is separable (with respect to δ), if there exists a countable dense subset T_0 of T such that for each t in T , $X(t) = \lim_{T_0 \ni s \rightarrow t} X(s)$, almost surely. By Proposition 2.1

$$\|P.(t) - P.(s)\|_{m_{J, 2q}} \leq \mathcal{B} d(s, t), \quad \|P.(s)\|_{m_{J, 2q}} \leq \mathcal{B} \left(\sum_{n=1}^N |c_n|^2 \right)^{1/2}.$$

The trajectories $s \mapsto P_\theta(s)$ being continuous for every θ , $P.$ is thus trivially separable. As $d^2(s, t) \leq 4\pi^2 |s - t|^2 \sum_{n=1}^N |c_n|^2 (\varphi_n^2 \wedge \frac{1}{\pi^2 |s-t|^2})$, we deduce that

$$(2.17) \quad d(s, t) \leq 2\pi |s - t| \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2},$$

once $\pi |s - t| \leq 1/\tilde{\varphi}_N$. Consider a covering $\{I_j, j = j_1, \dots, j_1 + H\}$ of L with intervals

$$I_j = \left[\frac{j-1}{\pi \tilde{\varphi}_N}, \frac{j}{\pi \tilde{\varphi}_N} \right[\quad (j \geq 1).$$

Introduce an auxiliary process \mathcal{Y} defined for $s \in I_j, j \geq 1$ by

$$\mathcal{Y}_s = \frac{P.(s) - P.(\frac{j-1}{\pi \tilde{p}_N})}{2\pi \mathcal{B} \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2}}.$$

By (2.17), for every $s, t \in I_j$

$$(2.18) \quad \|\mathcal{Y}_s - \mathcal{Y}_t\|_{m_{J, 2q}} = \frac{\|P.(s) - P.(t)\|_{m_{J, 2q}}}{2\pi \mathcal{B} \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2}} \leq \frac{d(s, t)}{2\pi \mathcal{B} \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2}} \leq |s - t|.$$

Thus $\{\mathcal{Y}_s, s \in I_j\}$ satisfies (2.14) with the usual metric. Recall that m_{I_j} denotes the normalized Lebesgue measure on I_j . Then

$$\int_0^{\text{diam}(I_j)} \frac{d\varepsilon}{m_{I_j}(B(s, \varepsilon))^{1/2q}} \leq \int_0^{1/(\pi \tilde{p}_N)} \left(\frac{1}{\pi \tilde{p}_N \varepsilon} \right)^{1/2q} d\varepsilon = \frac{1}{\pi \tilde{p}_N} \int_0^1 \eta^{-1/2q} d\varepsilon \leq \frac{c_q}{\tilde{p}_N}.$$

Hence

$$\sup_{s \in I_j} \int_0^{\text{diam}(I_j)} \frac{d\varepsilon}{m_{I_j}(B(s, \varepsilon))^{1/2q}} \leq \frac{c_q}{\tilde{p}_N}.$$

From (2.16) follows that

$$(2.19) \quad \sup_{j=1}^N \left\| \sup_{s,t \in I_j} |\mathcal{Y}_s - \mathcal{Y}_t| \right\|_{m_J, 2q} \leq \frac{c'_q}{\tilde{p}_N}.$$

Assume that $|L|\pi\tilde{\varphi}_N > 1$, and let $\{I_j, j = j_1, \dots, j_1 + H\}$, $H \geq 0$, be a covering of L . Let $s \in L$, and let j be such that $s \in I_j$. By writing

$$P(s) = P\left(\frac{j-1}{\pi\tilde{p}_N}\right) + \left(P(s) - P\left(\frac{j-1}{\pi\tilde{p}_N}\right)\right) = P\left(\frac{j-1}{\pi\tilde{p}_N}\right) + 2\pi\mathcal{B}\left(\sum_{n=1}^N |c_n|^2 \varphi_n^2\right)^{1/2} \mathcal{Y}_s,$$

next using the triangle inequality, we get

$$(2.20) \quad \begin{aligned} \left\| \sup_{s \in L} |P(s)| \right\|_{m_J, 2q} &\leq \left\| \sup_{1 \leq j \leq H} \left| P\left(\frac{j-1}{\pi\tilde{p}_N}\right) \right| \right\|_{m_J, 2q} \\ &\quad + 2\pi\mathcal{B}\left(\sum_{n=1}^N |c_n|^2 \varphi_n^2\right)^{1/2} \left\| \sup_{\substack{1 \leq j \leq H \\ s \in I_j}} |\mathcal{Y}_s| \right\|_{m_J, 2q}. \end{aligned}$$

In the one hand

$$(2.21) \quad \left\| \sup_{1 \leq j \leq H} \left| P\left(\frac{j-1}{\pi\tilde{p}_N}\right) \right| \right\|_{m_J, 2q} \leq H^{\frac{1}{2q}} \sup_{1 \leq j \leq H} \left\| P\left(\frac{j-1}{\pi\tilde{p}_N}\right) \right\|_{m_J, 2q} \leq \mathcal{B}H^{\frac{1}{2q}} \left(\sum_{n=1}^N |c_n|^2 \right)^{\frac{1}{2}}.$$

And in the other

$$(2.22) \quad \left\| \sup_{\substack{1 \leq j \leq H \\ s \in I_j}} |\mathcal{Y}_s| \right\|_{m_J, 2q} \leq H^{\frac{1}{2q}} \sup_{1 \leq j \leq H} \left\| \sup_{s \in I_j} |\mathcal{Y}_s| \right\|_{m_J, 2q}.$$

But $\mathcal{Y}\left(\frac{j-1}{\pi\tilde{\varphi}_N}\right) = 0$, and so by (2.19)

$$\left\| \sup_{s \in I_j} |\mathcal{Y}_s| \right\|_{m_J, 2q} \leq \left\| \sup_{s,t \in I_j} |\mathcal{Y}_s - \mathcal{Y}_t| \right\|_{m_J, 2q} \leq \frac{c'_q}{\tilde{p}_N}.$$

As $H \leq C \max(1, |L|\tilde{\varphi}_N)$, we deduce

$$(2.23) \quad \left\| \sup_{s \in L} |P(s)| \right\|_{m_J, 2q} \leq C_q \mathcal{B}(|L|\tilde{\varphi}_N)^{\frac{1}{2q}} \left\{ \left(\sum_{n=1}^N |c_n|^2 \right)^{1/2} + \frac{1}{\tilde{\varphi}_N} \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2} \right\}.$$

Finally, if $|L|\pi\tilde{\varphi}_N \leq 1$, write $L = [L_1, L_2]$. Given $s, t \in L$, we have $\pi|s-t| \leq \pi|L| \leq 1/\tilde{\varphi}_N$, and so

$$\|P(s) - P(t)\|_{m_J, 2q} \leq \mathcal{B}d(s, t) \leq 2\pi\mathcal{B}|s-t| \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2}.$$

Put

$$\mathcal{P}_s = \frac{P(s) - P(L_1)}{2\pi\mathcal{B}\left(\sum_{n=1}^N |c_n|^2 \varphi_n^2\right)^{1/2}}, \quad s \in L.$$

Then $\|\mathcal{P}_s - \mathcal{P}_t\|_{m_J, 2q} \leq |s-t|$. Similarly as for getting (2.19), we obtain

$$(2.24) \quad \left\| \sup_{s,t \in L} |\mathcal{P}_s - \mathcal{P}_t| \right\|_{m_J, 2q} \leq c_q |L|.$$

It follows that

$$(2.25) \quad \left\| \sup_{s \in L} |P(s)| \right\|_{m_J, 2q} \leq C'_q \mathcal{B} \left\{ \left(\sum_{n=1}^N |c_n|^2 \right)^{1/2} + |L| \left(\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right)^{1/2} \right\}.$$

With (2.23) and (2.15), we arrived to

$$\left\| \sup_{s \in L} |P(s)| \right\|_{m_J, 2q} \leq C''_q \mathcal{B} \max\{1, |L|\tilde{\varphi}_N\}^{\frac{1}{2q}} \left\{ \left[\sum_{n=1}^N |c_n|^2 \right]^{\frac{1}{2}} \right\}$$

$$(2.26) \quad + \min \left(|L|, \frac{1}{\tilde{\varphi}_N} \right) \left[\sum_{n=1}^N |c_n|^2 \varphi_n^2 \right]^{\frac{1}{2}} \Big\}.$$

This achieves the proof. \square

3. Proof of Theorem 1.1

The constants appearing in Turán's result (Section 1) are important. We have therefore explicated all constants appearing in our proof.

We begin with applying Theorem 2.3 to

$$P(N_1, N_2, t) = \sum_{N_1 \leq p \leq N_2} p^{-it}$$

where $N \leq N_1 < N_2 \leq 2N$. We have $\tilde{\varphi}_N \leq \sup\{\log p, p \leq 2N\} \leq C \log N$ and by using (2.13),

$$(3.1) \quad \mathcal{B} \leq \left(q! \left[1 + \frac{2\pi p_{N_1}^q}{|J|} \right] \right)^{1/2q} \leq C_q \max \left(1, \frac{N^q}{|J|} \right)^{1/2q}.$$

Let L be such that $|L| \geq 1$. Since $\pi(2x) - \pi(x) \leq \frac{x}{\log x}$ for any integer $x > 1$, we have $\pi(N_2) - \pi(N_1) \leq \pi(2N) - \pi(N) < N/\log N$, we have

$$\sum_{N \leq p \leq 2N} \log^2 p \leq \log^2(2N) \sum_{N \leq p \leq 2N} 1 \leq \frac{N \log^2(2N)}{\log N} \leq CN \log N.$$

We get

$$(3.2) \quad \begin{aligned} \left\| \sup_{t \in L} |P(N_1, N_2, t)| \right\|_{m_J, 2q} &\leq C_q \max \left(1, \frac{N^q}{|J|} \right)^{1/2q} (|L| \log N)^{1/2q} \left\{ \left(\frac{N}{\log N} \right)^{1/2} \right. \\ &\quad \left. + \frac{1}{\log N} \left(\sum_{N \leq p \leq 2N} \log^2 p \right)^{1/2} \right\}. \\ &\leq C_q \left(\max \left(1, \frac{N^q}{|J|} \right) |L| \log N \right)^{1/2q} \left(\frac{N}{\log N} \right)^{1/2}. \end{aligned}$$

So that if $|J| \leq N^q$,

$$(3.3) \quad \left\| \sup_{t \in L} |P(N_1, N_2, t)| \right\|_{m_J, 2q} \leq C_q \frac{N}{(\log N)^{1/2}} \left(\frac{|L| \log N}{|J|} \right)^{1/2q}.$$

The remaining part of the proof now consists of carefully adjusting the parameters in order to apply Turán's result (1.1).

Main parameters: $(\mathbf{H}, \delta, \mathbf{q}, \mathbf{B}, \nu, \mathbf{m}, \alpha)$. The constants H, δ, q, α are numerical and fixed. They will produce the constant c in (1.1). See (3.14).

Let $H \geq 2$ be some integer. Put

$$\delta = \frac{H-1}{8H} \quad q = \frac{5}{1-8\delta} = 5H.$$

Then

$$0 < \delta < 1/8 \quad \text{and} \quad q > \frac{4(\delta+1)}{1-8\delta}.$$

In addition we set

$$B = 4q\delta + 2(\delta+1),$$

and notice that $2B = 8q\delta + 4(\delta+1) < q$.

Now fix some positive integer ν and set

$$U = 2^\nu, \quad J = [U^{2B}, 2U^{2B}], \quad L = [U^B, 8U^B].$$

Let $N = 2^m$ with $m \geq \nu$. It follows that $|J| = U^{2B} \leq U^q \leq N^q$. Then

$$(3.4) \quad \left\| \sup_{2^m \leq N_1 < N_2 \leq 2^{m+2}} \sup_{t \in L} |P.(N_1, N_2, t)| \right\|_{m_J, 2q} \leq C_q \frac{2^{m(1+1/q)}}{m^{1/2}} \left(\frac{|L|m}{|J|} \right)^{1/2q}.$$

By Minkowski's inequality

$$\begin{aligned} & \left\| \sup_{\nu \leq m \leq \nu(1+\delta)} \sup_{2^m \leq N_1 < N_2 \leq 2^{m+2}} \sup_{t \in L} |P.(N_1, N_2, t)| \right\|_{m_J, 2q} \\ & \leq \left\| \sum_{\nu \leq m \leq \nu(1+\delta)} \sup_{2^m \leq N_1 < N_2 \leq 2^{m+2}} \sup_{t \in L} |P.(N_1, N_2, t)| \right\|_{m_J, 2q} \\ & \leq C_q \left(\frac{|L|}{|J|} \right)^{1/2q} \sum_{\nu \leq m \leq \nu(1+\delta)} 2^{m(1+1/q)} m^{1/2q-1/2} \\ & \leq C_q \nu^{1/2q-1/2} \left(\frac{|L|}{|J|} \right)^{1/2q} \sum_{\nu \leq m \leq \nu(1+\delta)} 2^{m(1+1/q)} \\ & \leq 2C_q \nu^{1/2q-1/2} 2^{-(B/2q)\nu} 2^{\nu(1+\delta)(1+1/q)}. \end{aligned}$$

Now if $U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta}$, choose $\nu \leq m \leq \nu(1+\delta)$ such that $2^m \leq N < 2^{m+1}$. Then $2^m \leq N \leq N_1 < N_2 \leq 2N < 2^{m+2}$. Thus

$$\begin{aligned} & \left\| \sup_{U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta}} \sup_{t \in L} |P.(N_1, N_2, t)| \right\|_{m_J, 2q} \\ & \leq \left\| \sup_{\nu \leq m \leq \nu(1+\delta)} \sup_{2^m \leq N_1 < N_2 \leq 2^{m+2}} \sup_{t \in L} |P.(N_1, N_2, t)| \right\|_{m_J, 2q} \\ & \leq 2C_q 2^{\nu[(1+\delta)(1+1/q)-(B/2q)]} \nu^{1/2q-1/2} \\ (3.5) \quad & \leq 2C_q 2^{[1-\delta]\nu} \nu^{1/2q-1/2} := M. \end{aligned}$$

since with our choices $(1+\delta)(1+1/q) - B/2q = 1 - \delta$.

Next let $0 < \alpha < 1$ be fixed and set $\mu(\alpha) = 1/(1-\alpha)^{1/(2q)}$. Set

$$\tilde{J} = \left\{ \theta \in J : \sup_{\substack{U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta} \\ t \in L}} |P_\theta(N_1, N_2, t)| \leq \mu(\alpha)M \right\}.$$

By the Tchebycheff inequality

$$\begin{aligned} \frac{1}{|J|} \lambda\{J \setminus \tilde{J}\} & \leq \frac{1}{|J|(\mu M)^{2q}} \int_J \sup_{\substack{U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta} \\ t \in L}} |P_\theta(N_1, N_2, t)|^{2q} d\theta \\ (3.6) \quad & \leq \mu(\alpha)^{-2q} = 1 - \alpha. \end{aligned}$$

Therefore $\lambda\{\tilde{J}\} \geq \alpha|J|$ and for all $\theta \in \tilde{J}$,

$$(3.7) \quad \sup_{\substack{U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta} \\ t \in L}} |P_\theta(N_1, N_2, t)| \leq 2\mu(\alpha)C_q 2^{[1-\delta]\nu} \nu^{1/2q-1/2}.$$

Pick some θ in \tilde{J} . Then

$$\begin{aligned} \sup_{\substack{U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta} \\ \tau \in \theta + L}} \left| \sum_{N_1 \leq p \leq N_2} \frac{1}{p^{i\tau}} \right| & \leq 2\mu(\alpha)C_q 2^{\nu(1-\delta)} \nu^{1/2q-1/2} \\ & = 2\mu(\alpha)C_q U^{1-\delta} (\log U)^{1/2q-1/2} \\ (3.8) \quad & \leq 2\mu(\alpha)C_q \frac{U (\log U)^{1/2q-1/2}}{U^\delta}. \end{aligned}$$

But if $\tau \in \theta + L$, $\tau \leq 2U^{2B} + 8U^B \leq 3U^{2B}$ if U , namely ν is large enough. It follows that $U^\delta \geq C\tau^{\delta/(2B)}$.

Put

$$b := \frac{\delta}{2B} = \frac{\delta}{8q\delta + 4(\delta + 1)}.$$

We have obtained:

For all $\tau \in [\theta + U^B, \theta + 8U^B]$ and $U \leq N \leq N_1 < N_2 \leq 2N \leq U^{1+\delta}$,

$$(3.9) \quad \left| \sum_{N_1 \leq p \leq N_2} \frac{1}{p^{i\tau}} \right| \leq 2\mu(\alpha)C_q \frac{N(\log N)^{1/2q-1/2}}{\tau^b}.$$

A family of local zerofree regions: We use secondary parameters: δ_0, D, b . Let

$$T = T_\theta = \theta + 3\sqrt{\theta}.$$

We may assume $\theta \geq 1$. In the one hand

$$T - \sqrt{T} = \theta + 3\sqrt{\theta} - \sqrt{\theta}\sqrt{1 + 3/\sqrt{\theta}} \geq \theta + 3\sqrt{\theta} - 2\sqrt{\theta} = \theta + \sqrt{\theta} \geq \theta + U^B.$$

And in the other since $U^{2B} \leq \theta \leq 2U^{2B}$

$$T + \sqrt{T} = \theta + 3\sqrt{\theta} + \sqrt{\theta}\sqrt{1 + 3/\sqrt{\theta}} \leq \theta + 5\sqrt{\theta} \leq \theta + 5\sqrt{2}U^B \leq \theta + 8U^B.$$

Hence $[T - \sqrt{T}, T + \sqrt{T}] \subset \theta + L$ and estimate (3.9) is valid for $T - \sqrt{T} \leq \tau \leq T + \sqrt{T}$. Further, as

$$U^{2B} \leq \theta \leq T = \theta + 3\sqrt{\theta} \leq 2U^{2B} + 3\sqrt{2}U^B = U^{2B}[2 + 3\sqrt{2}U^{-B}] \leq 7U^{2B},$$

it is also valid in the restricted range of values

$$(3.10) \quad T^{\frac{1}{2B}} \leq N \leq N_1 < N_2 \leq 2N \leq \left(\frac{T}{7}\right)^{\frac{1+\delta}{2B}}.$$

Now select a positive real δ_0 such that

$$0 < \frac{2\delta_0}{1-\delta_0} < \delta.$$

We notice that $1 + \delta - \frac{1+\delta_0}{1-\delta_0} = \delta - \frac{2\delta_0}{1-\delta_0} > 0$. Choose ν sufficiently large so that $2^{\nu[\delta - \frac{2\delta_0}{1-\delta_0}]} \geq 7^{1+\delta}$. Since $2B > 1$ we have

$$T^{1+\delta - \frac{1+\delta_0}{1-\delta_0}} \geq 2^{2B\nu(1+\delta - \frac{1+\delta_0}{1-\delta_0})} \geq 2^{\nu[1+\delta - \frac{1+\delta_0}{1-\delta_0}]} = 2^{\nu[\delta - \frac{2\delta_0}{1-\delta_0}]} \geq 7^{1+\delta},$$

namely

$$\left(\frac{T}{7}\right)^{1+\delta} \geq T^{\frac{1+\delta_0}{1-\delta_0}}.$$

Next put

$$D = \frac{1}{2B(1-\delta_0)}.$$

Then (3.10) implies the admissibility of the more suitable field of parameters

$$(3.11) \quad T^{D(1-\delta_0)} = T^{\frac{1}{2B}} \leq N \leq N_1 < N_2 \leq 2N \leq T^{D(1+\delta_0)} = T^{\frac{1+\delta_0}{2B(1-\delta_0)}} \leq \left(\frac{T}{7}\right)^{\frac{1+\delta}{2B}}.$$

Estimate (3.9) then implies

$$(3.12) \quad \left| \sum_{N_1 \leq p \leq N_2} \frac{1}{p^{i\tau}} \right| \leq 2\mu(\alpha)C_q \frac{N(\log N)^{1/2q-1/2}}{\tau^b},$$

for all $\tau \in [T - T^{1/2}, T + T^{1/2}]$ and all $T^{D(1-\delta_0)} \leq N \leq N_1 < N_2 \leq 2N \leq T^{D(1+\delta_0)}$.

Recall that $0 < \delta < 1/8$ and $q = \frac{5}{1-8\delta}$. Thus

$$B = 4q\delta + 2(\delta + 1) < \frac{20\delta}{1-8\delta} + \frac{9}{4} = \frac{80\delta + 9 - 72\delta}{4(1-8\delta)} = \frac{8\delta + 9}{4(1-8\delta)} < \frac{5}{2(1-8\delta)}.$$

And

$$b = \frac{\delta}{2B} \geq \frac{\delta(1-8\delta)}{5}.$$

In order that $b^{1/6} \geq \delta_0$, it suffices that $\frac{\delta(1-8\delta)}{5} \geq (\delta/2)^6$, namely $1 - 8\delta \geq (5/2^6)\delta^5$, which is fulfilled if $\delta < 1/9$ for instance, namely recalling that $\delta = \frac{H-1}{8H}$, if $H < 9$ which we do.

Thus $b \geq \delta_0^6$ does hold, and (3.12) implies that the inequality

$$(3.13) \quad \left| \sum_{N_1 \leq p \leq N_2} \frac{1}{p^{i\tau}} \right| \leq c \frac{N(\log N)^{1/2q-1/2}}{\tau^{\delta_0^6}},$$

with (recalling that $\mu(\alpha) = 1/(1 - \alpha)^{1/(2q)}$),

$$(3.14) \quad c = 2\mu(\alpha)C_q,$$

holds for all $\tau \in [T - T^{1/2}, T + T^{1/2}]$ and all $T^{D(1-\delta_0)} \leq N \leq N_1 < N_2 \leq 2N \leq T^{D(1+\delta_0)}$.

Turán's result (section 1) then implies that

$$(3.15) \quad \zeta(\sigma + it) \neq 0, \quad \forall \sigma > 1 - \delta_0^{12}, \quad \forall t \in [T_\theta - T_\theta^{1/2}, T_\theta + T_\theta^{1/2}].$$

But this holds for *any* $\theta \in \tilde{J}$ (recalling that $\lambda(\tilde{J}) \geq \alpha|J|$, $J = [2^{2B\nu}, 2^{2B\nu+1}]$), and for *any* ν , assuming this one large enough, depending on δ , say ν_δ . We also recall that δ was fixed from the beginning (see "Main parameters").

Remark 3.1. Finding *one* θ in J such that $\zeta(\sigma + it) \neq 0$ for all t in $[T_\theta - T_\theta^{1/2}, T_\theta + T_\theta^{1/2}]$ and $\sigma > \sigma_0$, for some $\sigma_0 < 1$, can be deduced from Carlson's estimate on the number of zeros of the Riemann zeta function. The point here is that we have a measurable set of values of θ 's of measure close to the one of J , for which this is valid. This together with a simple covering argument will permit to exhibit a much bigger zerofree zone.

A semi-global zerofree region: Let $\psi(\theta) = \theta + 3\sqrt{\theta}$. The indice ν with $\nu \geq \nu_\delta$ being now temporarily fixed, let $J_0 =]2^{2B\nu}, 2^{2B\nu+1}[\setminus \tilde{J}$. Using the fact that $\lambda(\psi([a, b])) = (b - a) + 3(\sqrt{b} - \sqrt{a}) \leq (b - a)\{1 + 2.2^{-B\nu}\}$, one can show

$$(3.16) \quad \lambda(\psi(J_0)) \leq \{1 + 1/2^{B\nu}\}(1 - \alpha)\lambda(J).$$

Let $\eta > 0$, J_0 being an open set, $J_0 = \cup_{n=1}^\infty I_n$ where I_n are open intervals, Let $U_N = \cup_{n=1}^N I_n$. Writing $U = U_N \cup B$ with $B \subset \cup_{n=N+1}^\infty I_n$, we have,

$$\begin{aligned} \lambda(\psi(J_0)) &\leq \lambda(\psi(U_N) \cup \psi(B)) \leq \lambda(\psi(U_N)) + \sum_{n=N+1}^\infty \lambda(\psi(I_n)) \\ &\leq \lambda(\psi(U_N)) + \{1 + 2.2^{-B\nu}\} \sum_{n=N+1}^\infty \lambda(I_n) \leq \lambda(\psi(U_N)) + \eta\{1 + 2.2^{-B\nu}\}, \end{aligned}$$

assuming N large enough. Further $\cup_{n=1}^N I_n = \cup_{n=1}^{N'} I'_n$, I'_n being pairwise disjoint intervals. Since ψ is continuous increasing,

$$\begin{aligned} \lambda(\psi(U_N)) &= \lambda\left(\sum_{n=1}^{N'} \psi(I'_n)\right) = \sum_{n=1}^{N'} \lambda(\psi(I'_n)) \leq \{1 + 2.2^{-B\nu}\} \sum_{n=1}^N \lambda(I_n) \\ &= \{1 + 2.2^{-B\nu}\} \lambda(U_N) \leq \{1 + 2.2^{-B\nu}\} (\lambda(J_0) + \eta). \end{aligned}$$

Thus

$$\lambda(\psi(J_0)) \leq \{1 + 1/2^{B\nu}\} \lambda(J_0) + 2\eta\{1 + 1/2^{B\nu}\} \leq \{1 + 1/2^{B\nu}\} \{ (1 - \alpha)\lambda(J) + 2\eta \},$$

since $\lambda(J_0) \leq (1 - \alpha)\lambda(J)$. Since η is arbitrary, (3.16) follows.

Therefore,

$$\begin{aligned} \lambda(\psi(\tilde{J})) &\geq \lambda(\psi(J)) - \{1 + 2^{-B\nu}\}(1 - \alpha)\lambda(J) \\ &= \lambda(\psi(J)) \left[1 - \frac{1 + 2^{-B\nu}}{1 + 3(\sqrt{2} - 1)2^{-B\nu}} (1 - \alpha) \right] \end{aligned}$$

$$(3.17) \quad := (1 - \bar{\alpha})\lambda(\psi(J)),$$

noticing that $\lambda(\psi(J)) = \lambda(J)(1 + 3(\sqrt{2} - 1)2^{-B\nu})$.

As $T_\theta^{1/2} \geq \theta^{1/2} \geq 2^{B\nu}$, we have $[T_\theta - T_\theta^{1/2}, T_\theta + T_\theta^{1/2}] \supset [T_\theta - 2^{B\nu}, T_\theta + 2^{B\nu}]$. Now consider on $\psi(J) = [\psi(2^{2B\nu}), \psi(2^{2B\nu+1})]$ the subdivision

$$K_i = \left[\psi(2^{2B\nu}) + (i-1)2^{B\nu-1}, \psi(2^{2B\nu}) + i2^{B\nu-1} \right], \quad 1 \leq i \leq (2^{B\nu+1} + 6(\sqrt{2} - 1)).$$

In view of (3.17), the number of indices i such that $K_i \cap \psi(\tilde{J}) = \emptyset$ is less than $(1 - \bar{\alpha})\lambda(\psi(J))/2^{B\nu+1}$.

Consequently, at least $\bar{\alpha}\lambda(\psi(J))/2^{B\nu+1}$ indices i are such that $K_i \cap \psi(\tilde{J}) \neq \emptyset$. Pick a real ϑ in the intersection. We have

$$[\vartheta - \vartheta^{1/2}, \vartheta + \vartheta^{1/2}] \supset K_i.$$

So that by (3.15),

$$(3.18) \quad \zeta(\sigma + it) \neq 0, \quad \forall \sigma > 1 - \delta_0^{12}, \quad \forall t \in K_i,$$

and the number of indices i for which this is true, exceeds

$$(3.19) \quad \bar{\alpha}\lambda(\psi(J))/2^{B\nu+1} = \bar{\alpha}(2^{B\nu+1} + 6(\sqrt{2} - 1)).$$

We can now achieve the proof. Given any fixed real $0 < \alpha^* < 1$, it follows from (3.18), (3.19) that in any subdivision of $\psi(J)$ of size $2^{B\nu-1}$, at least $\alpha^*2^{B\nu+1}$ intervals are free of zero. Since $\psi(J) = [2^{2B\nu} + 3 \cdot 2^{B\nu}, 2 \cdot 2^{2B\nu} + 3\sqrt{2} \cdot 2^{B\nu}]$, it also implies that in any subdivision of $[2^{2B\nu}, 2^{2B\nu+1}]$ of size $2^{B\nu-1}$, at least $\alpha^*2^{B\nu+1}$ intervals are free of zero.

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